Eigenvalues and Eigenvectors

Subject: Advanced Linear Algebra

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Dr. Mahesh Kumar Sharma
Assiatant Professor
School of Basic and Applied Sciences

Eigenvalue

- Let A be a real or complex square matrix of order n as

$$A = [a_{ij}]_{n\times n} \quad \text{where} \quad i, j = 1, 2, 3......, n$$

- The characteristic polynomial of A is defined as

$$\pi(\lambda) = det(A - \lambda I_n),$$

  where I_n is an identity matrix of order n.

- A number λ (may be real or complex) is said to be an **eigenvalue** of matrix A if and only if it is a root of its characteristic polynomial.

Example

- Example 1: Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Solution: By definition, the characteristic polynomial is

$$\pi(\lambda) = det(A - \lambda I_2) = det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 \qquad (1)$$

- The above polynomial has two roots viz. $+1$ and -1, which are the required eigenvalues.

Example

- Example 2: Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Solution: By definition, the characteristic polynomial is

$$\pi(\lambda) = det(A - \lambda I_2) = det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \qquad (2)$$

- The above polynomial has two roots viz. $+i$ and $-i$, which are the required eigenvalues.

Example

- Example 3: Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

- Solution: By definition, the characteristic polynomial is

$$\pi(\lambda) = det(A - \lambda I_2) = det \left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= det \begin{bmatrix} -\lambda & i \\ -i & -\lambda \end{bmatrix} = \lambda^2 - 1 \qquad (3)$$

- The above polynomial has two roots viz. $+1$ and -1, which are the required eigenvalues.

Eigenvector

- A non-zero vector **x** (having order $n \times 1$) which satisfy the relation

$$A\mathbf{x} = \lambda\mathbf{x}, \tag{4}$$

is said to be an **eigenvector** of A.

- The above relation (2) can be re-written as

$$(A - \lambda I)\mathbf{x} = \mathbf{0}, \tag{5}$$

Eigenvector in case of Example 1

- To calculate the eigenvectors $\mathbf{X} = [x_1 \quad x_2]^T$ corresponding to eigenvalue $\lambda_1 = 1$, we need to solve

$$(A - I)\mathbf{X} = \mathbf{0}, \tag{6}$$

This gives

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Example Cont...

which further can be written as

$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned} \tag{8}$$

- By solving (8), we obtain $x_1 = x_2$,. If we select $x_1 = 1$, we get $x_2 = 1$. Thus the eigenvector corresponding to $\lambda_1 = 1$ is $\mathbf{X} = [1 \quad 1]^T$.
- To calculate the eigenvectors $\mathbf{X} = [x_1 \quad x_2]^T$ corresponding to eigenvalue $\lambda_1 = -1$, we need to solve

$$(A + I)\mathbf{X} = \mathbf{0}, \tag{9}$$

Example Cont...

This gives

$$\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + 1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Or

$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{10}$$

which further can be written as

$$\begin{aligned}x_1 + x_2 &= 0\\x_1 + x_2 &= 0\end{aligned} \tag{11}$$

- By solving (11), we obtain $x_1 = -x_2$,. If we select $x_1 = 1$, we get $x_2 = -1$. Thus the eigenvector corresponding to $\lambda_1 = -1$ is $\mathbf{X} = [1 \quad -1]^T$.